0. Preprocess input

1. Recursively solve one/many subproblems (same problem on a smaller instance)

2. Use their answers to solve for given input

CSE525 Lec4: Recursion

Debajyoti Bera (M21)

3 2 14 6 10 | 5.5 19 1 13 5

1 2
5 3
5.5 6
13 10
19 14

Select(A,k)

Naive solution: Sorting based

bad run

cn ←	n
$c(n-2)$	$n-2$
$c(n-4)$	$n-4$
⋮	⋮

$\Theta(n)$ recursive calls

A [$<p$ | p | $>p$]
1 2 ⋯ t-1 t

- p is the t-th smallest elmt in A
- if k=t, return p
- if k<t, QuickSel(A[1⋯t-1], k)

Given an array A[1…n] and an integer k <= n, return the k-*th* small element in A.

QuickSelect(A,k): $=n/2$

Won't work

Quick
pivot=Select(A,n/2)

1. If A is small, brute force

1.5

2. r = partition(A) // returns rank(pivot)
3. If k < r: return QuickSelect(A,k) A[1⋯r-1]
4. If k = r: return A[k]
5. If k > r: return QuickSelect(A, __) A[r+1⋯n]

best way to choose pivot is median= Select(A,n/2)

Time complexity recurrence:

T(n) = ____________________________

T(small) = brute force

T(n) = O(n) // time to partition

\+ _____________ // time during recursion

What is the best way to choose pivot ?

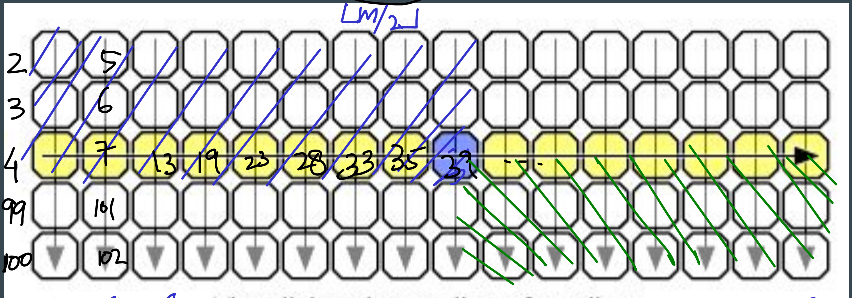
Alternative 1.5.

MomSelect(A, K) : // |A| = n
0. basecase
1.
2. r = partition (A, pivot = mom)
3. if K < r : return MomSelect(A[1...r-1], K)
4. 5, as before, using MomSelect(...)

Median of (5) Median

Pivot selection

$m \leftarrow \lceil n/5 \rceil$
for $i \leftarrow 1$ to m ← Brute force/naive
  $M[i] \leftarrow$ MEDIANOFFIVE$(A[5i-4..5i])$ ⟨⟨Brute force!⟩⟩
$mom \leftarrow$ MOMSELECT$(M[1..m], \lfloor m/2 \rfloor)$ ⟨⟨Recursion!⟩⟩



Visualizing the median of medians

Use median of medians (mom) as pivot !

Lemma: At least 3 elements of subarray A[5i-4 ... 5i] are <= mom.

n = 100 median has rank 50
mom has rank [30, 31, ..., 70]

Lemma: At least 3*(n/5)/2 elements of A are <= mom.

rank(x) = #(elements ≤ x)

Lemma: At least 3n/10 elements of A are > mom.

$(\#\text{elmts} < \text{mom}) \leq n - \frac{3n}{10} = \frac{7n}{10}$

Thm: 3n/10 <= rank(pivot) <= 7n/10

$\therefore \text{rank(mom)} < \frac{7n}{10}$

Q: Final recurrence ?

- Find median of slots
- Find median of medians
- Recursively solve using mom as pivot.

$(\#\text{element} > \text{mom}) \geq 3\lceil m/2 \rceil = 3\lceil \lceil n/5 \rceil / 2 \rceil \geq \frac{3n}{10}$

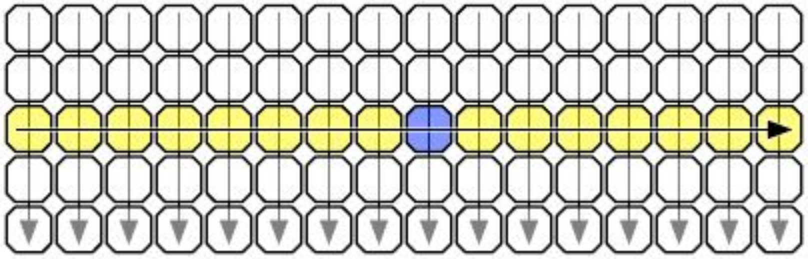
Q: Solve recurrence.

$(\#\text{elements} \leq \text{mom}) \geq 3\lfloor m/2 \rfloor = 3\lfloor \lceil n/5 \rceil / 2 \rfloor \geq 3n/10$

$\therefore \text{rank(mom)} \geq \frac{3n}{10}$

Median of (5) Median

$m \leftarrow \lceil n/5 \rceil$
for $i \leftarrow 1$ to m
    $M[i] \leftarrow \textsc{MedianOfFive}(A[5i-4\,..\,5i])$ ⟨⟨*Brute force!*⟩⟩
$mom \leftarrow \textsc{MomSelect}(M[1\,..\,m], \lfloor m/2 \rfloor)$ ⟨⟨*Recursion!*⟩⟩



Visualizing the median of medians

Use median of medians (mom) as pivot !

Thm: 3n/10 <= rank(pivot) <= 7n/10

Groups of Three

$T(n) \le O(n) + T(\frac{n}{3}) + T(\frac{2n}{3})$

$\text{rank}(\text{pivot}) \in \{\frac{n}{3}, \frac{2n}{3}\}$

33% 67%

Q: Final recurrence ?

- Find median of slots
- Find median of medians
- Recursively solve using mom as pivot.

Q: Solve recurrence.

Quick

$T(\dots)$ is an increasing fn

Select(A,k) using MOM

$\frac{3n}{10} \le r \le \frac{7n}{10}$ $\qquad$ $\frac{7n}{10} \ge n-r \ge \frac{3n}{10}$

Use median of medians (mom) as pivot ! 3n/10 <= rank(pivot) <= 7n/10

MomSelect(A,k):

1. If A is small, brute force
2. M = ComputeMOM(A) //pivot selection
3. r = partition(A,M) // returns rank(pivot)
4. If k < r: return ~~Quick~~ Mom Select(~~A~~,k) A[1 ... r-1]
5. If k = r: return A[k]
6. If k > r: return ~~Quick~~ Mom Select(~~A~~, __) A[r+1 ... n]

computing median of M[i] $\qquad$ partition

$$T(n) = \left\lceil \frac{n}{5} \right\rceil * O(1) + T\left(\left\lceil \frac{n}{5} \right\rceil\right) + O(n) + \underbrace{\max\{T(r), T(n-r)\}}_{\le T\left(\frac{7n}{10}\right)}$$

$$\le O(n) + T(n/5) + T\left(\frac{7n}{10}\right)$$

$$= O(n) \quad \text{conv. series} \quad \left\{ \frac{n}{5} + \frac{7n}{10} = \frac{9n}{10} < n \right.$$

$n \rightarrow cn$

$\frac{n}{5}, \frac{7n}{10} \rightarrow c\frac{9}{10}n$

$\frac{n}{25}, \frac{7n}{50}, \frac{7n}{50}, \frac{49n}{100} \rightarrow c \cdot \frac{81}{100} n = c\left(\frac{9}{10}\right)^2 n$

Solving Recurrence

$$T(n) <= O(n) + T(n/5) + T(7n/10)$$

$$T(n) <= O(n) + max\_\{t:0...n-1\}\ T(t)$$

Why median of 5 ?

Practical implementations

- Quickselect (1961) in most libraries (with pivot selection heuristics)
- IntroSelect : Perform Quickselect and observe behaviour. If Quickselect appears slow, switch to IntroSelect
- Martinez + (2002) : Switch to sorting when array size becomes small
- Floyd Rivest randomized selection : 2 pivots and 3 partitions
- Chen (2015) : Groups of 2, 3, 4 are possible (needs to be done cleverly)
- Alexandrescu (2017) : Faster in practice. Uses median of 3 medians of three.

Diameter(G) = $\text{longest}_{u,v}$ shortest path(u,v)

(Diameter of a Tree, Ht of a tree)

left dia = 20
rt dia = 20
diameter of H = 32

max(left dia, right dia, 2 + left ht + right ht)

Recursively compute …

Diameter (v) = diameter of subtree rooted at v